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The Bernstein Constant and Polynomial Interpolation at the Chebyshev Nodes

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In this paper, we establish new asymptotic relations for the errors of approximation in $L_p[-1,1]$, $0<p\leqslant\infty$, of $|x|^\lambda$, $\lambda>0$, by the Lagrange interpolation polynomials at the Chebyshev nodes of the first and second kind. As a corollary, we show that the Bernstein constant

$$B_{\lambda,p} := \lim_{n\to\infty} n^{\lambda+1/p} \inf_{c_k} \| \, |x|^\lambda - \sum_{k=0}^{n} c_k x^k \|_{L_p[-1,1]}$$

is finite for $\lambda>0$ and $p\in(\frac{1}{3},\infty)$. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

Let $\mathscr{P}_n$ be the set of all algebraic polynomials of degree at most n with real coefficients; B_1 the class of all entire functions of exponential type 1 with real coefficients; $L_p(\Omega)$ the quasi-normed space of measurable real-valued functions f on $\Omega\subseteq\mathbf{R}$ with the finite quasi-norm $\|f\|_{L_p(\Omega)} := (\int_\Omega |f|^p\,dx)^{1/p}$, $0<p<\infty$; $L_\infty(\Omega)$ the space of all continuous real-valued functions f on $\Omega\subseteq\mathbf{R}$ with the finite norm $\|f\|_{L_\infty(\Omega)} := \sup_\Omega|f|$; $L_p[a,b] := L_p([a,b])$, $0<p\leqslant\infty$.

It was Bernstein [3] who in 1938 initiated the study of polynomial approximation to $f_\lambda(t) := |t|^\lambda$ in the uniform metric by proving the following result: for any $\lambda>0$ there exists a constant $B_{\lambda,\infty}\in(0,\infty)$ such that

$$\lim_{n\to\infty} n^\lambda \min_{P_n\in\mathscr{P}_n} \|f_\lambda - P_n\|_{L_\infty[-1,1]} = B_{\lambda,\infty}. \quad (1.1)$$

The case $\lambda=1$ was investigated earlier in [1]. The proofs in [1, 3] are difficult, and many non-trivial technical details are missing. This is the reason why some mathematicians have doubts about accuracy of the proofs. In particular, V. M. Tikhomirov in 1988 and S. M. Nikolskii in 2000 asked

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the author as to whether limit relation (1.1) was valid. The answer was affirmative since there is a rigorous and fairly short proof of (1.1). In 1946 Bernstein [4, 23, p. 48] came up with the idea of general limit relations between polynomial and harmonic approximations which imply (1.1), see [5, 23, p. 416] for details. A more precise limit theorem was obtained by the author [9]: for any continuous f of polynomial growth on $\mathbf{R}$,

$$\lim_{n\to\infty} \min_{P_n\in\mathscr{P}_n} ||f - P_n||_{L_\infty[-n+\delta_n, n-\delta_n]} = \inf_{g\in B_1} ||f - g||_{L_\infty(\mathbf{R})}, \tag{1.2}$$

where $\delta_n = \sqrt{n}$, $n = 1, 2, \ldots$. It immediately follows from (1.2) that (1.1) holds true with

$$B_{\lambda,\infty} = \inf_{g\in B_1} ||f_\lambda - g||_{L_\infty(\mathbf{R})}, \tag{1.3}$$

(see [10, 11]). Note that the Jackson-type theorem for approximation of continuous functions of polynomial growth by entire functions of exponential type (cf. [5, 23, pp. 257–259]) implies that the right-hand side of (1.3) is finite, and so Bernstein's result can be reformulated as follows: if the Bernstein constant $B_{\lambda,\infty}$ is defined by (1.3), then $B_{\lambda,\infty} < \infty$ and limit relation (1.1) holds.

The problem of finding $B_{\lambda,\infty}$ is still open and seems very difficult. Using high-precision calculations, Varga and Carpenter [22] computed $B_{1,\infty} = 0.28017 + \alpha$, where $|\alpha| \leqslant 4\times 10^{-6}$.

Raitsin [15] showed that the uniform norm in (1.2) can be replaced by L_p-norm, $1 \leqslant p < \infty$, provided that $\delta_n = 0$ and $\inf_{g\in B_1} ||f - g||_{L_p(\mathbf{R})} < \infty$. The author [11] extended this result to $p \in (0, 1)$. As a corollary, we have the following L_p-analogue of Bernstein's result, if

$$B_{\lambda,p} := \inf_{g\in B_1} ||f_\lambda - g||_{L_p(\mathbf{R})}, \qquad 0 < p < \infty, \quad \lambda > -1/p, \tag{1.4}$$

is finite, then

$$\lim_{n\to\infty} n^{\lambda+1/p} \min_{P_n\in\mathscr{P}_n} ||f_\lambda - P_n||_{L_p[-1,1]} = B_{\lambda,p}. \tag{1.5}$$

The explicit expression for the Bernstein constant $B_{\lambda,p}$ is known only for $p = 1$ [6, 11, 14] and $p = 2$ [16],

$$B_{\lambda,1} = (8/\pi)\Gamma(\lambda+1)|\sin(\pi\lambda/2)| \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-\lambda-2}, \qquad \lambda > -1,$$

$$B_{\lambda,2} = (2/\sqrt{\pi})\Gamma(\lambda+1)|\sin(\pi\lambda/2)|\sqrt{2\lambda+1}, \qquad \lambda > -\tfrac{1}{2}. \tag{1.6}$$

The problem of finding necessary and sufficient conditions on $\lambda > -1/p$ and $p \in (0, \infty)$ for $B_{\lambda,p} < \infty$ was posed in [12].

In this paper we discuss analogues of (1.1) and (1.5) for Lagrange interpolation to f_λ at the Chebyshev nodes of the first and second kind. The corresponding constants will be effectively evaluated, and like (1.3) and (1.4) they can be expressed via the error of approximation by the interpolation functions from B_1. As a corollary, we obtain a partial solution to the problem from [12].

Let $S_{2n,i} \in \mathscr{P}_{2n}$ be the unique Lagrange interpolation polynomial to $|t|^\lambda$ on $[-1,1]$ at the Chebyshev nodes $\{t_{k,i}\}$ of the ith kind, $i = 1,2,\ \lambda > 0$. Here, $t_{0,i} = 0,\ \ i = 1,2$ and

$$t_{k,1} = \cos\frac{(k-\frac{1}{2})\pi}{2n}, \quad k = 1,\dots,2n,$$

$$t_{k,2} = \cos\frac{k\pi}{2n+2}, \quad k = 1,\dots,n,n+2,\dots,2n+1.$$

The A_N-approximation error is defined by

$$L_{\lambda,p,i}(N,A_N) := ||f_\lambda - S_{N,i} - (-1)^n N^{-\lambda} A_N\, T_{N,i}||_{L_p[-1,1]},$$
$$0<p\leqslant\infty, \quad i = 1,2,$$

where $N = 2n,\ n \in \mathbf{N}$; A_N is a real constant; and $T_{N,1}(t) := T_N(t)$, $T_{N,2}(t) := U_{N+1}(t)/(Nt)$. Here

$$T_m(t) := (\tfrac{1}{2})((t+\sqrt{t^2-1})^m + (t-\sqrt{t^2-1})^m),$$

$$U_m(t) := (\tfrac{1}{2})((t+\sqrt{t^2-1})^{m+1} - (t-\sqrt{t^2-1})^{m+1})/\sqrt{t^2-1}$$

are the Chebyshev polynomials of the first and second kind, respectively.

Approximation properties of the Lagrange interpolation polynomials to f_λ have attracted much attention in the 1990s and 2000s [7, 8, 13, 17–21]. In particular, Revers [17] proved that for $N = 2,4,\dots$ and $\lambda \in (0,\frac{2}{3}] \cup \{1\}$, the following estimate holds:

$$L_{\lambda,\infty,1}(N,0) \leqslant 2(\tfrac{2}{3})^{1-\lambda} N^{-\lambda}.$$

Here we study the asymptotic behavior of $L_{\lambda,p,i}(N,A_N)$ for $0<p\leqslant\infty$ and $i = 1,2$, as $N \to \infty$.

Notation. Throughout the paper λ is a real number, $\lambda \neq 0,2,\dots$, and $C, C_1,\dots$ denote positive constants independent of N, n, t, z, x, M, ε, B, φ. The same symbol does not necessarily denote the same constant in different occurrences. Let us set

$$\gamma_1 := \tfrac{1}{2}, \qquad \gamma_2 := \tfrac{1}{3}, \qquad \varphi_1(t) := \cos t, \qquad \varphi_2(t) := \sin t/t.$$

We also make use of the following functions and constants for $\lambda > 0$:

$$F_{\lambda,1}(t) := (4/\pi)\sin(\pi\lambda/2)\int_0^\infty \frac{y^{\lambda-1}}{(1+(y/t)^2)(e^y+e^{-y})}\,dy,$$

$$F_{\lambda,2}(t) := (4/\pi)\sin(\pi\lambda/2)\int_0^\infty \frac{y^{\lambda}}{(1+(y/t)^2)(e^y-e^{-y})}\,dy,$$

$$C_1(\lambda) := (4/\pi)\sin(\pi\lambda/2)\int_0^\infty \frac{y^{\lambda-1}}{e^y+e^{-y}}\,dy$$
$$= (4/\pi)\sin(\pi\lambda/2)\Gamma(\lambda)\sum_{k=0}^\infty (-1)^k(2k+1)^{-\lambda},$$

$$C_2(\lambda) := (4/\pi)\sin(\pi\lambda/2)\int_0^\infty \frac{y^{\lambda}}{e^y-e^{-y}}\,dy$$
$$= (4/\pi)\sin(\pi\lambda/2)\Gamma(\lambda+1)\sum_{k=0}^\infty (2k+1)^{-(\lambda+1)},$$

$$\Phi_{\lambda,1}(t) := C_1(\lambda) - F_{\lambda,1}(t) = (4/\pi)\sin(\pi\lambda/2)\int_0^\infty \frac{y^{\lambda+1}}{(t^2+y^2)(e^y+e^{-y})}\,dy, \tag{1.7}$$

$$\Phi_{\lambda,2}(t) := C_2(\lambda) - F_{\lambda,2}(t) = (4/\pi)\sin(\pi\lambda/2)\int_0^\infty \frac{y^{\lambda+2}}{(t^2+y^2)(e^y-e^{-y})}\,dy. \tag{1.8}$$

It is easy to see that $|F_{\lambda,i}|$ is increasing in $t \in (0,\infty)$. Next, using the fact that $\Phi_{\lambda,i}(0) = C_i(\lambda)$ and $\lim_{t\to\infty}\Phi_{\lambda,i}(t) = 0$ for $i = 1,2$ (see also Lemma 3(a)), we arrive at

$$|\Phi_{\lambda,i}(0)| = |C_i(\lambda)| = \lim_{t\to\infty}|F_{\lambda,i}(t)| = ||F_{\lambda,i}||_{L_\infty(\mathbf{R})}, \qquad i = 1,2. \tag{1.9}$$

Finally, we define the following special sequences:

$$A_{2n,1} = A_{2n,1}(\lambda) := (2/\pi)\sin(\pi\lambda/2)n^{\lambda} \int_1^{\infty} \frac{(u-1)^{\lambda-1}(u+1)}{u^{\lambda/2+1}(u^n+u^{-n})}\,du,$$

$$A_{2n,2} = A_{2n,2}(\lambda) := (1/\pi)\sin(\pi\lambda/2)n^{\lambda+1} \int_1^{\infty} \frac{(u-1)^{\lambda}(u+1)^2}{u^{\lambda/2+2}(u^{n+1}-u^{-(n+1)})}\,du.$$

We shall show in the next section (see Lemma 1) that

$$\lim_{N=2n\to\infty} A_{N,i}(\lambda) = C_i(\lambda), \qquad i=1,2. \tag{1.10}$$

2. STATEMENT OF MAIN RESULTS

Let $P := [\lambda/2]$ and let

$$g_{\lambda,A,1}(t) := \cos t\left(A + \sum_{l=0}^{P-1} C_1(\lambda-2l-2)t^{2(l+1)} + 2t^{2(P+1)}\sum_{k=0}^{\infty}(-1)^{k+1}\frac{((k+\frac{1}{2})\pi)^{\lambda-2P-1}}{t^2-((k+\frac{1}{2})\pi)^2}\right),$$

$$g_{\lambda,A,2}(t) := \sin t\left(A + \sum_{l=0}^{P-1} C_2(\lambda-2l-2)t^{2l+1} + 2t^{2P+1}\sum_{k=1}^{\infty}(-1)^{k}\frac{(k\pi)^{\lambda-2P}}{t^2-(k\pi)^2}\right),$$

be the entire functions of exponential type 1 that interpolate f_λ at the nodes $\{(k+\frac{1}{2})\pi\}_{k=-\infty}^{\infty}$ and $\{k\pi\}_{|k|=1}^{\infty}$, respectively. We shall show in Section 4 (see Lemma 5) that $g_{\lambda,A,i}$ are the unique even interpolating functions from B_1 to f_λ that satisfy the conditions $f_\lambda - g_{\lambda,A,i} \in L_\infty(\mathbf{R})$ and $g_{\lambda,A,i}(0) = A$, $i=1,2$.

First, we discuss the asymptotics for $L_{\lambda,p,1}$.

THEOREM 1. *Let* $\lambda > 0$, $\lambda \neq 2,4,\ldots$ *and* $A \in \mathbf{R}$.

(a) *If* $\lim_{n\to\infty} A_{2n} = A$, *then*

$$\lim_{N=2n\to\infty} N^{\lambda}L_{\lambda,\infty,1}(N, A_N) = \|f_\lambda - g_{\lambda,A,1}\|_{L_\infty(\mathbf{R})} = \max(|C_1(\lambda)-A|, |A|). \tag{2.1}$$

(b) *If* $\lim_{n\to\infty} A_{2n} = A$ *and* $0<|t|\leqslant 1$, *then*

$$\limsup_{N=2n\to\infty} N^{\lambda}\, |f_\lambda(t) - S_{N,1}(t) - (-1)^n N^{-\lambda} A_N\, T_{N,1}(t)| = |C_1(\lambda) - A|. \tag{2.2}$$

(c) *If* $p \in (\frac{1}{2}, \infty)$, *then*

$$\begin{aligned}\lim_{N=2n\to\infty} N^{\lambda+1/p} L_{\lambda,p,1}(N, A_{N,1}) &= ||f_\lambda - g_{\lambda,C_1(\lambda),1}||_{L_p(\mathbf{R})} \\ &= ||\varphi_1 \Phi_{\lambda,1}||_{L_p(\mathbf{R})} < \infty. \end{aligned}\tag{2.3}$$

(d) *Let* $\lim_{n\to\infty} A_{2n} = A,\ A \in \mathbf{R}$. *If* $p \in (0,\infty),\ A \neq C_1(\lambda)$ *or* $p \in (0, \frac{1}{2}]$, $A = C_1(\lambda)$, *then*

$$\lim_{N=2n\to\infty} N^{\lambda+1/p} L_{\lambda,p,1}(N, A_N) = \infty. \tag{2.4}$$

The similar asymptotics hold true for $L_{\lambda,p,2}(N, A_N)$.

THEOREM 2. *Let* $\lambda > 0$ *and* $\lambda \neq 2, 4, \ldots$

(a) *If* $p \in (\frac{1}{3}, \infty]$, *then*

$$\begin{aligned}\lim_{N=2n\to\infty} N^{\lambda+1/p} L_{\lambda,p,2}(N, A_{N,2}) &= ||f_\lambda - g_{\lambda,C_2(\lambda),2}||_{L_p(\mathbf{R})} \\ &= ||\varphi_2 \Phi_{\lambda,2}||_{L_p(\mathbf{R})} < \infty. \end{aligned}\tag{2.5}$$

(b) *Let* $\lim_{n\to\infty} A_{2n} = A,\ A \in \mathbf{R}$. *If* $p \in (0,1],\ A \neq C_2(\lambda)$ *or* $p \in (0, \frac{1}{3}]$, $A = C_2(\lambda)$, *then*

$$\lim_{N=2n\to\infty} N^{\lambda+1/p} L_{\lambda,p,2}(N, A_N) = \infty. \tag{2.6}$$

For some special cases it is possible to give a more explicit form to the expressions in Theorems 1 and 2.

COROLLARY 1. *For* $\lambda > 0$, *the following asymptotics hold*:

$$\lim_{N=2n\to\infty} N^{\lambda} L_{\lambda,\infty,1}(N, 0) = \lim_{N=2n\to\infty} N^{\lambda} L_{\lambda,\infty,1}(N, C_1(\lambda)) = |C_1(\lambda)|, \tag{2.7}$$

$$\min \lim_{N=2n\to\infty} N^{\lambda} L_{\lambda,\infty,1}(N, A_N) = \lim_{N=2n\to\infty} N^{\lambda} L_{\lambda,\infty,1}(N, C_1(\lambda)/2)$$
$$= |C_1(\lambda)|/2, \tag{2.8}$$

$$\lim_{N=2n\to\infty} N^{\lambda+1} L_{\lambda,1,1}(N, A_{N,1}) = 2|C_1(\lambda+2)|/(\lambda+1), \tag{2.9}$$

$$\lim_{N=2n\to\infty} N^{\lambda} L_{\lambda,\infty,2}(N, A_{N,2}) = |C_2(\lambda)|, \tag{2.10}$$

where min *in* (2.8) *is taken over all convergent sequences* $\{A_{2n}\}_{n=1}^{\infty}$.

Note that relations (2.7) and (2.8) immediately follow from (2.1), and (2.10) follows from (2.5), while (2.9) is an easy consequence of (2.3) and the formula $||\varphi_1 \Phi_{\lambda,1}||_{L_1(\mathbf{R})} = 2|C_1(\lambda+2)|/(\lambda+1)$ [5, 12].

We remark that the estimate $\limsup_{N=2n\to\infty} N^{\lambda} L_{\lambda,\infty,1}(N,0) \leqslant |C_1(\lambda)|$ was given in [2, p. 100].

Since (2.3)–(2.5), (2.8), and (2.10) imply the inequality

$$B_{\lambda,p} \leqslant C_{\lambda,p} := \begin{cases} \min(|C_1(\lambda)|/2, |C_2(\lambda)|), & p=\infty, \\ \min\limits_i ||\varphi_i \Phi_{\lambda,i}||_{L_p(\mathbf{R})}, & p \in (\frac{1}{3}, \infty), \end{cases} \tag{2.11}$$

we immediately arrive at the following result.

COROLLARY 2. *For* $\lambda > 0$ *and* $p \in (\frac{1}{3}, \infty]$, *relation* (1.5) *holds true with the finite Bernstein constant* $B_{\lambda,p}$ *satisfying* (2.11).

Remark 1. Combining (2.9) with (1.6) and (2.11), we conclude that for $\lambda > 0$, $B_{\lambda,1} = C_{\lambda,1}$, and polynomials $S_{2n,1} + (-1)^n (2n)^{-\lambda} A_{2n,1} T_{2n}$ are asymptotically best approximations to f_λ in $L_1[-1,1]$ (see also [11, 14]). It seems plausible that $B_{\lambda,p} < C_{\lambda,p}$ for $p \neq 1$.

The proofs of Theorems 1 and 2 are based on the following asymptotics.

LEMMA 1. (a) *For* $N=2n$, $n \in \mathbf{N}$, $\lambda > 0$, *and* $t \in [-1,1]$, *we have*

$$|t|^{\lambda} - S_{N,i}(t) = (-1)^n N^{-\lambda} T_{N,i}(t)(F_{\lambda,i}(Nt)(1+\alpha_{N,i,1}(t)) + \beta_{N,i,1}(t)), \tag{2.12}$$

$$|t|^{\lambda} - S_{N,i}(t) - (-1)^n N^{-\lambda} A_{N,i} T_{N,i}(t)$$
$$= (-1)^{n+1} N^{-\lambda} T_{N,i}(t)(\Phi_{\lambda,i}(Nt)(1+\alpha_{N,i,2}(t)) + \beta_{N,i,2}(t)), \tag{2.13}$$

where $|\alpha_{N,i,j}(t)| \leqslant CN^{-1/3}$ *and* $|\beta_{N,i,j}(t)| \leqslant CN^{\lambda}\exp(-C_1 N^{1/3})$, $i = 1, 2$, $j = 1, 2$.

(b) *If* $\lambda > 0$, *then* (1.10) *holds.*

Remark 2. Note that the proof of (1.1) in [3] was based on a weaker version of (2.12) for $i = 1$. The proof of this asymptotic was outlined in [2].

3. PROOF OF LEMMA 1

The proof follows [2, pp. 92, 98–100], though we added some technical details missing in [2].

We first need the following result.

LEMMA 2. *Let* $P_{n,i} \in \mathscr{P}_n$ *be the Lagrange interpolation polynomial to* $(1-x)^s$ *on* $[-1, 1]$ *at the nodes* $x_{0,i} = 1$,

$$x_{k,i} = \begin{cases} \cos\frac{(k-1/2)\pi}{n}, & i = 1, \\ \cos\frac{k\pi}{n+1}, & i = 2, \end{cases} \qquad k = 1, \ldots, n,$$

where $n > s > 0$. *Then for any* $x \in [-1, 1]$,

$$(1-x)^s - P_{n,i}(x) = (1/\pi)\sin\pi s\,(1-x)Q_{n,i}(x) \times \int_1^\infty \frac{(z-1)^{s-1}}{(z-x)Q_{n,i}(z)}\,dz, \qquad i = 1, 2, \tag{3.1}$$

where

$$Q_{n,i}(z) = \begin{cases} T_n(z), & i = 1, \\ U_n(z), & i = 2. \end{cases}$$

Proof. Let $P_{n,j,a} \in \mathscr{P}_n$ be the interpolation polynomial to $(a-x)^s$ on $[-1, 1]$ at $\{x_{k,j}\}_{k=0}^n$, where $a > 1$ and $j = 1, 2$. By the Hermite error formula for Lagrange interpolation [25],

$$(a-x)^s - P_{n,j,a}(x) = \frac{R_j(x)}{2\pi i}\lim_{M\to\infty}\lim_{\varepsilon\to 0}\int_{D_{M,\varepsilon}}\frac{(a-z)^s}{(z-x)R_j(z)}\,dz, \tag{3.2}$$

where $R_j(x) := (x-1)Q_{n,j}(x)$ and $(a-z)^s$ takes positive values for real $z < a$, $s > 0$. Here, $D_{M,\varepsilon} = C_{M,\varepsilon} \cup C_\varepsilon \cup D_\varepsilon \cup D_{-\varepsilon}$ is a contour in $\mathbf{C}$, oriented in a positive sense, where M and ε, $M > a > (a-1)/2 > \varepsilon > 0$, are fixed numbers and

$$C_{M,\varepsilon} := \{z : |z| = M, \arcsin(\varepsilon/M) \leqslant |\arg z| \leqslant \pi\},$$

$$C_\varepsilon := \{z : |z-a| = \varepsilon,\ \pi/2 \leqslant |\arg z| \leqslant \pi\},$$

$$D_{\pm\varepsilon} := \{z = x \pm i\varepsilon : a \leqslant x \leqslant \sqrt{M^2 - \varepsilon^2}\}.$$

Since the function $h_j(z) := \frac{(a-z)^s}{(z-x)R_j(z)}$ satisfies the conditions

$$\max_{z \in C_{M,\varepsilon}} |h_j(z)| \leqslant CM^{s-n-2}, \qquad \max_{z \in C_\varepsilon} |h_j(z)| \leqslant C\varepsilon^s,$$

we have

$$\lim_{M\to\infty} \lim_{\varepsilon\to 0} \int_{C_{M,\varepsilon}} h_j(z)\,dz = \lim_{M\to\infty} \lim_{\varepsilon\to 0} \int_{C_\varepsilon} h_j(z)\,dz = 0. \tag{3.3}$$

Next, by the limit relation

$$\lim_{\varepsilon\to 0} ((a-(x+i\varepsilon))^s - (a-(x-i\varepsilon))^s = -2i \sin \pi s (x-a)^s, \qquad x \geqslant a,$$

we obtain

$$\lim_{M\to\infty} \lim_{\varepsilon\to 0} \left(\int_{D_\varepsilon} h_j(z)\,dz + \int_{D_{-\varepsilon}} h_j(z)\,dz \right) = -2i \sin \pi s \int_a^\infty \frac{(z-a)^s}{(z-x)R_j(z)}\,dz. \tag{3.4}$$

Then (3.2)–(3.4) yield the integral representation

$$(a-x)^s - P_{n,j,a}(x) = (1/\pi) \sin \pi s (1-x) Q_{n,j}(x) \int_a^\infty \frac{(z-a)^s}{(z-x)(z-1)Q_{n,j}(z)}\,dz. \tag{3.5}$$

Finally, letting $a \to 1+$ in (3.5) and taking account of the relation $\lim_{a\to 1+} P_{n,j,a}(x) = P_{n,j}(x)$, we obtain (3.1). ∎

Proof of Lemma 1. (a) By the substitution $z = (\frac{1}{2})(u + u^{-1})$, we have

$$\begin{aligned} I_{n,1}(x) &:= (1-x) \int_1^\infty \frac{(z-1)^{s-1}}{(z-x)T_n(z)}\,dz \\ &= 2^{2-s}(1-x) \int_1^\infty \frac{(u-1)^{2s-1}(u+1)}{u^s(1+u^2-2ux)(u^n+u^{-n})}\,du \\ &= 2^{2-s}\left((1-x) \int_1^{1+n^{-2/3}} + (1-x) \int_{1+n^{-2/3}}^\infty\right) \\ &= 2^{2-s}(I_1(x) + I_2(x)). \end{aligned} \tag{3.6}$$

Then since

$$\max_{x\in[-1,1]} \frac{1-x}{1+u^2-2ux} \leqslant \frac{2}{(1+u)^2}, \qquad u \geqslant 0,$$

we obtain for $n > s$

$$\begin{aligned} I_2(x) &\leqslant 2 \int_{1+n^{-2/3}}^\infty \frac{(u-1)^{2s-1}\,du}{u^{s+n}(1+u)} \leqslant \frac{2(1+n^{-2/3})^{-(n-s)}}{n-s+1} \\ &\leqslant C\exp(-n^{1/3}). \end{aligned} \tag{3.7}$$

Next by the substitution $u = 1 + y/n$, we have

$$I_1(x) = n^{-2s} \int_0^{n^{1/3}} \frac{y^{2s-1}(1+\frac{y}{2n})}{(1+\frac{y}{n})^s(1+\frac{y}{n}+\frac{y^2}{2(1-x)n^2})((1+\frac{y}{n})^n + (1+\frac{y}{n})^{-n})}\,dy. \tag{3.8}$$

Further, it is easy to see that for all $y \in [0, n^{1/3}]$ and $n \geqslant 1$,

$$e^y \geqslant \left(1+\frac{y}{n}\right)^n \geqslant e^{y-y^2/(2n)} \geqslant e^{y-1/(2n^{1/3})} \geqslant (1-1/(2n^{1/3}))e^y,$$

$$e^{-y} \leqslant \left(1+\frac{y}{n}\right)^{-n} \leqslant (1+n^{1/3})e^{-y}.$$

Hence we have for $y \in [0, n^{1/3}]$ and $n \geqslant 1$

$$\begin{aligned} (1-n^{-1/3})(e^y+e^{-y}) &\leqslant (1+y/n)^n + (1+y/n)^{-n} \\ &\leqslant (1+n^{-1/3})(e^y+e^{-y}). \end{aligned}$$

Then it follows from these inequalities and (3.8) that

$$\begin{aligned} I_1(x) &= (1+\gamma_{n,1}(x))n^{-2s} \int_0^{n^{1/3}} \frac{y^{2s-1}}{(1+\frac{y^2}{2(1-x)n^2})(e^y+e^{-y})}\,dy \\ &= (1+\gamma_{n,1}(x))n^{-2s} \int_0^{\infty} \frac{y^{2s-1}}{(1+\frac{y^2}{2(1-x)n^2})(e^y+e^{-y})}\,dy + \mu^*_{n,1}(x), \end{aligned} \tag{3.9}$$

where $|\gamma_{n,1}(x)| \leqslant Cn^{-1/3}$ and $|\mu^*_{n,1}(x)| \leqslant C\exp(-n^{1/3})$.

Combining (3.9) with (3.1), (3.6), and (3.7), we obtain for any $n \in \mathbf{N}$

$$\begin{aligned} &(1-x)^s - P_{n,1}(x) \\ &\quad = \frac{2^{2-s}}{\pi} \sin \pi s n^{-2s} T_n(x) \\ &\quad\quad \times \left((1+\gamma_{n,1}(x)) \int_0^{\infty} \frac{y^{2s-1}}{(1+\frac{y^2}{2(1-x)n^2})(e^y+e^{-y})}\,dy + n^{2s}\mu_{n,1}(x) \right), \end{aligned} \tag{3.10}$$

where $|\mu_{n,1}(x)| \leqslant C\exp(-n^{1/3})$.

Similarly,

$$\begin{aligned} &(1-x)^s - P_{n,2}(x) \\ &\quad = (1/\pi)\sin \pi s(1-x)U_n(x) \int_1^{\infty} \frac{(z-1)^{s-1}}{(z-x)U_n(z)}\,dz \\ &\quad = \frac{2^{1-s}}{\pi} \sin \pi s(1-x)U_n(x) \int_1^{\infty} \frac{(u-1)^{2s}(u+1)^2}{u^{s+1}(1+u^2-2ux)(u^{n+1}-u^{-(n+1)})}\,du \\ &\quad = \frac{2^{2-s}}{\pi} \sin \pi s n^{-2s-1} U_n(x) \\ &\quad\quad \times \left((1+\gamma_{n,2}(x)) \int_0^{\infty} \frac{y^{2s}}{(1+\frac{y^2}{2(1-x)n^2})(e^y-e^{-y})}\,dy + n^{2s+1}\mu_{n,2}(x) \right), \end{aligned} \tag{3.11}$$

where $|\gamma_{n,2}(x)| \leqslant Cn^{-1/3}$ and $|\mu_{n,2}(x)| \leqslant C\exp(-n^{1/3})$.

Finally, making the substitutions $1 - x = 2t^2,\ \lambda = 2s,\ N = 2n$, in (3.10) and (3.11) and taking account of the relations

$$T_n(1-2t^2) = (-1)^n\, T_{N,1}(t), \qquad P_{n,i}(1-2t^2) = 2^s S_{N,i}(t), \quad i = 1,2, \tag{3.12}$$

$$(1/n)\,U_n(1-2t^2) = (-1)^n U_{2n+1}(t)/(2nt) = (-1)^n T_{N,2}(t), \tag{3.13}$$

we arrive at (2.12).

To prove (2.13), we use the similar argument. We first note that for $x \in [-1,1]$ and $N = 2n$,

$$\begin{aligned}
&(1-x)^{\lambda/2} - P_{n,2}(x) - (1/n)2^{\lambda/2}N^{-\lambda}A_{n,2}\,U_n(x)\\
&\quad = -(2^{1-\lambda/2}/\pi)\sin(\pi\lambda/2)\,U_n(x)\\
&\qquad \times \int_1^\infty \frac{(u-1)^{\lambda+2}(u+1)^2/2}{u^{\lambda/2+2}((1-u)^2+2u(1-x))(u^{n+1}-u^{-(n+1)})}\,du\\
&\quad = -(2^{1-\lambda/2}/\pi)\sin(\pi\lambda/2)\,U_n(x)\left(\int_1^{1+n^{-2/3}} + \int_{1+n^{-2/3}}^\infty\right)\\
&\quad = -(2^{1-\lambda/2}/\pi)\sin(\pi\lambda/2)\,U_n(x)(J_1(x)+J_2(x)).
\end{aligned} \tag{3.14}$$

Then estimates like (3.7) hold if n is large enough,

$$J_2(x) \leqslant C\int_{1+n^{-2/3}}^\infty \frac{(u-1)^\lambda}{u^{\lambda/2}(u^{n+1}-1)}\,du \leqslant C\exp(-n^{1/3}). \tag{3.15}$$

Next, relations like (3.8) and (3.9) are valid,

$$\begin{aligned}
J_1(x) &= \frac{n^{-(\lambda+3)}}{1-x}\\
&\quad\times \int_0^{n^{1/3}} \frac{y^{\lambda+2}(1+\frac{y}{2n})^2}{(1+\frac{y}{n})^{\lambda/2+2}(1+\frac{y}{n}+\frac{y^2}{2(1-x)n^2})((1+\frac{y}{n})^{n+1}-(1+\frac{y}{n})^{-(n+1)})}\,dy\\
&= 2(1+\gamma_{n,2}(x))n^{-(\lambda+1)}\int_0^\infty \frac{y^{\lambda+2}}{(2(1-x)n^2+y^2)(e^y-e^{-y})}\,dy\\
&\quad + \mu_{n,2}(x),
\end{aligned} \tag{3.16}$$

where $|\gamma_{n,2}(x)| \leqslant Cn^{-1/3}$ and $|\mu_{n,2}(x)| \leqslant C\exp(-n^{1/3})$. Making the substitution $1-x = 2t^2$ and taking account of (3.12) and (3.13), we obtain (2.13) for $i = 2$ from (3.14) to (3.16). Similarly for $i = 1$.

(b) To prove (1.10), we use the same asymptotic technique.

$$n^{\lambda+1}\int_1^{\infty}\frac{(u-1)^{\lambda}(u+1)^2}{u^{\lambda/2+2}(u^{n+1}-u^{-(n+1)})}\,du = n^{\lambda+1}\int_1^{1+n^{-2/3}} +o(1)$$
$$=4\int_0^{\infty}\frac{y^{\lambda}}{e^y-e^{-y}}\,dy+o(1),$$

as $n\to\infty$. Thus (1.10) follows for $i=2$. Similarly for $i=1$. ∎

4. PROPERTIES OF $\Phi_{\lambda,i}$

To prove Theorems 1 and 2, we need some properties of the functions $\Phi_{\lambda,i},\ \lambda>0,\ i=1,2$, which are given in Lemmas 3 and 5. Recall that these functions are defined by (1.7) and (1.8).

LEMMA 3. *For $i=1,2$ the following statements hold*:

(a) $|\Phi_{\lambda,i}|$ *is a decreasing function in* $t\in(0,\infty)$ *and* $\Phi_{\lambda,i}(\infty):=\lim_{t\to\infty}\Phi_{\lambda,i}(t)=0$.

(b) *If* $p\in(\gamma_i,\infty)$, *then* $\varphi_i\Phi_{\lambda,i}\in L_p(\mathbf{R})$ *and* $\tau_i\Phi_{\lambda,i}\in L_p(\mathbf{R}\backslash(-1,1))$, *where* $\tau_1(t):=1$ *and* $\tau_2(t):=1/t$.

(c) *If* $p\in(\gamma_i,\infty)$, *then*

$$\lim_{N=2n\to\infty} N^{1/p}||T_{N,i}\Phi_{\lambda,i}(N\cdot)||_{L_p[-1,1]}=||\varphi_i\Phi_{\lambda,i}||_{L_p(\mathbf{R})}<\infty. \tag{4.1}$$

(d) *Let* $p\in(0,\infty)$ *if* $i=1$ *and* $p\in(0,1]$ *if* $i=2$. *Then for any convergent sequence* $\{B_{2n_k}\}_{k=1}^{\infty}$ *satisfying* $\lim_{k\to\infty}B_{2n_k}\neq 0$, *we have*

$$D_1:=\liminf_{k\to\infty}\int_{-2n_k}^{2n_k}|T_{2n_k,i}(t/(2n_k))((1+O(n_k^{-4/3}))\Phi_{\lambda,i}(t)-B_{2n_k})|^p\,dt=\infty.$$

(e) *If* $p\in(0,\gamma_i]$, *then for any sequence* $\{B_{2n}\}_{n=1}^{\infty}$ *satisfying* $\lim_{n\to\infty}B_{2n}=0$, *we have*

$$D_2:=\liminf_{N=2n\to\infty}\int_{-N}^{N}|T_{N,i}(t/N)((1+O(N^{-4/3}))\Phi_{\lambda,i}(t)-B_N)|^p\,dt=\infty. \tag{4.2}$$

In the proof of Lemma 3 we shall use the following properties of Chebyshev polynomials.

LEMMA 4. (a) *If* $N = 2n$, *then*

$$||T_{N,1}||_{L_\infty[-1,1]} = 1, \qquad \lim_{N=2n\to\infty} ||T_{N,2}||_{L_\infty[-1,1]} = 1; \tag{4.3}$$

$$||T_{N,1}||_{L_p[-1,1]} \geqslant C, \qquad p \in (0,\infty); \tag{4.4}$$

$$||T_{N,2}||_{L_p[-1,1]} \geqslant \begin{cases} CN^{-1}, & p \in (0,1), \\ C \ln N/N, & p = 1. \end{cases} \tag{4.5}$$

(b) *For* $i = 1, 2$,

$$\lim_{N=2n\to\infty} T_{N,i}(t/N) = \varphi_i(t), \tag{4.6}$$

uniformly in any interval $[-B, B]$.

Proof. (a) It is easy to verify that

$$\begin{aligned} ||T_{N,2}||_{L_\infty[-1,1]} &= (1/n) \max_{t\in[0,1]} |U_n(1-2t^2)| \\ &= (1/n)||U_n||_{L_\infty[-1,1]} = (n+1)/n. \end{aligned} \tag{4.7}$$

Next, for $p \in (0,\infty)$,

$$||T_{N,1}||^p_{L_p[-1,1]} = 2\int_0^{\pi/2} |\cos(2nt)|^p \sin t\, dt \geqslant C \int_{\pi/4}^{\pi/2} |\cos(2nt)|^p\, dt \geqslant C. \tag{4.8}$$

Further for $p \in (0,1)$,

$$\begin{aligned} ||T_{N,2}||^p_{L_p[-1,1]} &\geqslant CN^{-p} \int_{\pi/4}^{\pi/2} t^{-p} |\sin((2n+1)t)|^p\, dt \\ &\geqslant CN^{-1} \sum_{k=[n/2]+1}^{n} \int_{(4k+1)\pi/4}^{(4k+2)\pi/4} y^{-p} |\sin y|^p\, dy \\ &\geqslant CN^{-1} \sum_{k=[n/2]+1}^{n} (4k+1)^{-p} \geqslant CN^{-p}. \end{aligned} \tag{4.9}$$

Similarly for $p = 1$. Thus (4.7)–(4.9) yield (4.3)–(4.5), respectively.

(b) Limit relation (4.6) is a special case of the Mehler–Heine asymptotic [20, Theorem 8.1.1]. ∎

Proof of Lemma 3. (a) Since $(t^2+y^2)^{-1} \leqslant t^{-2}$, we obtain from (1.7) and (1.8) $|\Phi_{\lambda,i}(t)| \leqslant |C_i(\lambda+2)|t^{-2}$ for $t>1$ and $i=1,2$. Hence $\Phi_{\lambda,i}(\infty)=0$, $i=1,2$.

(b) It follows from the estimates

$$\int_{\mathbf{R}} |\varphi_i \Phi_{\lambda,i}|^p \, dt = 2\left(\int_0^1 + \int_1^\infty \right) \leqslant C\left(|F_{\lambda,i}(\infty)|^p + |F_{\lambda+2,i}(\infty)|^p \int_1^\infty t^{-p/\gamma_i} \, dt \right) < \infty,$$

that $\varphi_i \Phi_{\lambda,i} \in L_p(\mathbf{R})$ for $p \in (\gamma_i, \infty)$, $i=1,2$. Similarly $\Phi_{\lambda,1} \in L_p(\mathbf{R})$ for $p \in (\frac{1}{2}, \infty)$ and $\Phi_{\lambda,2}(t)/t \in L_p(\mathbf{R}-(-1,1))$ for $p \in (\frac{1}{3}, \infty)$.

(c) Note first that by statement (b), for any $\varepsilon>0$ there exists $B_0>0$ such that for all $B>B_0$, $\int_{|t|>B} |\tau_i \Phi_{\lambda,i}|^p \, dt < \varepsilon$, $i=1,2$. Then taking account of Lemmas 4(b) and 3(b), we obtain for any $B>B_0$

$$\begin{aligned} &\limsup_{N=2n\to\infty} N||T_{N,i}\Phi_{\lambda,i}(N\cdot)||^p_{L_p[-1,1]} \\ &\quad = \limsup_{N=2n\to\infty} \int_{-N}^{N} |T_{N,i}(t/N)\Phi_{\lambda,i}(t)|^p \, dt \\ &\quad \leqslant \limsup_{N=2n\to\infty} \left(\int_{-B}^{B} |T_{N,i}(t/N)\Phi_{\lambda,i}(t)|^p \, dt + \int_{|t|>B} |\tau_i \Phi_{\lambda,i}|^p \, dt \right) \\ &\quad \leqslant ||\varphi_i \Phi_{\lambda,i}||^p_{L_p[-B,B]} + \varepsilon < C. \end{aligned} \tag{4.10}$$

Next using (4.6) again, we have

$$\liminf_{N=2n\to\infty} ||T_{N,i}\Phi_{\lambda,i}(N\cdot)||^p_{L_p[-1,1]} \geqslant ||\varphi_i \Phi_{\lambda,i}||^p_{L_p[-B,B]}. \tag{4.11}$$

Finally, letting $B\to\infty$ in (4.10) and (4.11), we arrive at (4.1).

(d) Without loss of generality we may assume that $\lim_{N=2n\to\infty} B_N \neq 0$. It follows from (4.3) that

$$\int_{-N}^{N} |T_{N,i}(t/N)\Phi_{\lambda,i}(t)|^p \, dt = 2\left(\int_0^1 + \int_1^N \right) \leqslant C_1 + C_2 \int_1^N t^{-p/\gamma_i} \, dt. \tag{4.12}$$

Then by (4.4), (4.5), and (4.12),

$$\begin{aligned} D_1 &\geqslant \liminf_{N=2n\to\infty} \left(C|B_N|^p \int_{-N}^{N} |T_{N,i}(t/N)|^p \, dt - C_1 - C_2 \int_1^N t^{-p/\gamma_i} \, dt \right) \\ &\geqslant C_3 \liminf_{N=2n\to\infty} \delta_{N,i}, \end{aligned}$$

where

$$\delta_{N,i} = \begin{cases} N - C_4 N^{1-2p}, & i=1,\ p \in (0,\infty),\ p \neq \frac{1}{2}, \\ N - C_4 \ln N, & i=1,\ p=\frac{1}{2}, \\ N^{1-p} - C_4 N^{1-3p}, & i=2,\ p \in (0,1),\ p \neq \frac{1}{3}, \\ N^{2/3} - C_4 \ln N, & i=2,\ p=\frac{1}{3}, \\ \ln N - C_4 N^{-2}, & i=2,\ p=1. \end{cases}$$

This proves the statement.

(e) Let $\lim_{N=2n\to\infty} B_N = 0$. We first need the following elementary inequality: for any sequences $\alpha_n \to 0,\ b_n \to 0$, as $n \to \infty$ and every $a > 0$ there exists $n_0 = n_0(a)$ such that for all $n > n_0$,

$$|a(1+\alpha_n) - b_n| \geqslant (1 - 2|\alpha_n|)|a - |b_n||.$$

Then using this inequality for $\alpha_n = O(n^{-4/3}),\ b_n = B_{2n}$, and $a = \Phi_{\lambda,i}(B)$, where $B \in (1,\infty)$, we obtain from statement (a) and relation (4.6)

$$\begin{aligned} D_2 &\geqslant \liminf_{N=2n\to\infty} \int_0^B |T_{N,i}(t/N)((1+O(N^{-4/3}))\Phi_{\lambda,i}(t) - B_N)|^p\,dt \\ &\geqslant \liminf_{N=2n\to\infty} (1 - CN^{-4/3})^p \int_0^B |T_{N,i}(t/N)(\Phi_{\lambda,i}(t) - B_N)|^p\,dt \\ &\geqslant \liminf_{N=2n\to\infty} \int_0^B |T_{N,i}(t/N)(\Phi_{\lambda,i}(t) - \Phi_{\lambda,i}(B)|^p\,dt \\ &= \int_0^B |\varphi_i(t)(\Phi_{\lambda,i}(t) - \Phi_{\lambda,i}(B))|^p\,dt. \end{aligned} \tag{4.13}$$

Since for $t \in [\frac{1}{2}, B/2]$,

$$|\Phi_{\lambda,i}(t) - \Phi_{\lambda,i}(B)| \geqslant C\left(\frac{1}{t^2+1} - \frac{1}{B^2+1}\right) \geqslant C\frac{B^2}{B^2+1}t^{-2},$$

we have from (4.13) for $p \in (0, \gamma_i]$

$$\begin{aligned} D_2 &\geqslant C \lim_{B\to\infty} (B^2/(B^2+1))^p \int_{1/2}^{B/2} |t^{-2p}|\varphi_i(t)|^p\,dt = C\int_{1/2}^{\infty} t^{-2p}|\varphi_i(t)|^p\,dt \\ &\geqslant C\sum_{k=0}^{\infty} \int_{\pi/4+2k\pi}^{\pi/3+2k\pi} t^{-p/\gamma_i}\,dt \geqslant C\sum_{k=0}^{\infty} (8k+1)^{-p/\gamma_i} = \infty. \end{aligned}$$

This completes the proof of the lemma. ∎

LEMMA 5. *Let $\lambda > 0$ and $\lambda \neq 2, 4\ldots$*

(a) *For* $t \in \mathbf{R}$ *the following series expansions hold*:

$$\varphi_i(t)F_{\lambda,i}(t) = |t|^\lambda - g_{\lambda,0,i}(t), \qquad i = 1, 2. \tag{4.14}$$

(b) $g_{\lambda,A,i} = g_{\lambda,0,i} + A\varphi_i$ *is the unique even function from* B_1 *to* f_λ *that interpolates* f_λ *at nodes* $\{(k+\frac{1}{2})\pi\}_{k=-\infty}^{\infty}$ *for* $i = 1$ *and at* $\{k\pi\}_{|k|=1}^{\infty}$ *for* $i = 2$ *and satisfies the following conditions*: $f_\lambda - g_{\lambda,A,i} \in L_\infty(\mathbf{R})$ *and* $g_{\lambda,A,i}(0) = A,\ i = 1, 2.$

Proof. We first prove (4.14) for $0 < \lambda < 2$ (cf. [2, p. 101]) , that is

$$\cos tF_{\lambda,1}(t) = |t|^\lambda - 2t^2 \cos t \times \sum_{k=0}^{\infty} (-1)^{k+1}((k+\tfrac{1}{2})\pi)^{\lambda-1}/(t^2 - ((k+\tfrac{1}{2})\pi)^2), \tag{4.15}$$

$$(\sin t/t)F_{\lambda,2}(t) = |t|^\lambda - 2t \sin t \sum_{k=1}^{\infty} (-1)^k (k\pi)^\lambda/(t^2 - (k\pi)^2). \tag{4.16}$$

Setting

$$h_1(z) := \frac{z^{\lambda-1}}{(1+(z/t)^2)(e^z + e^{-z})}, \qquad h_2(z) = \frac{z^\lambda}{(1+(z/t)^2)(e^z - e^{-z})},$$

we have

$$\begin{aligned} F_{\lambda,j}(t) &= (2i/\pi)\exp(-i\pi\lambda/2)\left(\int_0^\infty h_j(z)\,dz - \exp(i\pi\lambda)\int_0^\infty h_j(z)\,dz\right) \\ &= (2i/\pi)\exp(-i\pi\lambda/2)\int_{-\infty}^{\infty} h_j(z)\,dz \\ &= (2i/\pi)\exp(-i\pi\lambda/2) \lim_{M=\mu_j\pi\to\infty}\ \lim_{\varepsilon\to 0} \int_{D_{M,\varepsilon}} h_j(z)\,dz, \qquad j = 1, 2, \end{aligned} \tag{4.17}$$

where $\mu_j = 2m + (1 + (-1)^j)/4$, $m = 1, 2, \ldots,$ $j = 1, 2,$ and $D_{M,\varepsilon} = C'_M \cup C'_\varepsilon \cup D'_\varepsilon$ is a contour in $\mathbf{C}$, oriented in a positive sense. Here

$$C'_\delta := \{z : |z| = \delta,\ 0 < \arg z < \pi\}, \qquad D'_\varepsilon := \{z = x + i0\colon\ \varepsilon \leqslant |x| \leqslant M\}.$$

To justify the last equality in (4.17), we first note that

$$\lim_{\varepsilon\to 0} \int_{C'_\varepsilon} h_j(z)\,dz = 0. \tag{4.18}$$

Next, if $z = x + iy,\ |z| = 2m\pi$, then

$$|e^z + e^{-z}| \geqslant \begin{cases} (\frac{1}{2})e^{|x|}, & |x| \geqslant 1 \\ |\cos y| e^{|x|}, & |x| < 1 \end{cases} \geqslant (\tfrac{1}{2})e^{|x|}.$$

Similarly, for $|z| = |x + iy| = (2m + \frac{1}{2})\pi$,

$$|e^z - e^{-z}| \geqslant \begin{cases} (\frac{1}{2})e^{|x|}, & |x| \geqslant 1 \\ |\sin y| e^{|x|}, & |x| < 1 \end{cases} \geqslant (\tfrac{1}{2})e^{|x|}.$$

Hence if $M = \mu_j \pi$ is large enough, then

$$|h_j(Me^{i\varphi})| \leqslant CM^{\lambda-2}\, e^{-\cos\varphi}, \qquad |\varphi| < \pi/2, \quad j = 1, 2.$$

Using Jordan's Lemma, we obtain for $\lambda \in (0, 2)$

$$\lim_{M=\mu_j\pi\to\infty} \int_{C'_M} h_j(z)\, dz = 0, \qquad j = 1, 2. \tag{4.19}$$

Thus (4.18) and (4.19) imply the last equality in (4.17). Evaluating now the integral $\int_{D_{M,\varepsilon}} h_j(z)\, dz$ by the Residue Theorem, we arrive at (4.15) and (4.16).

Next let $\lambda > 2$. Then setting $P = [\lambda/2]$, we have

$$\begin{aligned}\cos t F_{\lambda,1}(t) &= (4/\pi)\sin(\pi\lambda/2) t^2 \cos t \int_0^\infty \frac{y^{\lambda-3}}{(1 + (t/y)^2)(e^y + e^{-y})}\, dy \\ &= (4/\pi)\sin(\pi\lambda/2)\cos t \\ &\quad \times \left(\sum_{l=0}^{P-1} (-1)^l t^{2(l+1)} \int_0^\infty \frac{y^{\lambda-2l-3}}{e^y + e^{-y}}\, dy \right. \\ &\quad \left. + (-1)^P t^{2P} \int_0^\infty \frac{y^{\lambda-2P-1}}{(1 + (y/t)^2)(e^y + e^{-y})}\, dy \right).\end{aligned} \tag{4.20}$$

Since $0 < \lambda - 2P < 2$, we may apply (4.15) to the last integral in (4.20). Thus (4.14) for $i = 1$ follows. Similarly using (4.16) and an analogue of (4.20) for $(\sin t/t)F_{\lambda,2}(t)$, we obtain (4.14) for $i = 2$.

(b) Note first that $g_{\lambda,A,i} \in B_1$ (cf. [23, p. 181]), and $g_{\lambda,A,i}(0) = A$. Then by (4.14), $f_\lambda - g_{\lambda,A,i} \in L_\infty(\mathbf{R})$, and $(f_\lambda - g_{\lambda,A,i})(t) = 0$ if and only if t is a node. The uniqueness of $g_{\lambda,A,i}$ can be proved by the standard argument [23, p. 180]: for any function g^* with the similar properties, $(g_{\lambda,A,i} - g^*)/\varphi_i$ is an entire function from $L_\infty(\mathbf{R})$ and $(g_{\lambda,A,i} - g^*)(0) = 0$. By Liouville's Theorem, $g_{\lambda,A,i} = g^*$, and this proves the lemma. ∎

5. PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1. (a) Let $\lim_{n\to\infty} A_{2n} = A$. It follows from (2.12) that

$$N^{\lambda}|\,|t|^{\lambda} - S_{N,1}(t) - (-1)^n N^{-\lambda} A_N T_{N,1}(t)| = |T_{N,1}(t)||F_{\lambda,1}(Nt) - A_N| + o(1), \qquad N = 2n \to \infty. \tag{5.1}$$

Hence taking account of (1.9) and (4.3), we have

$$\limsup_{N=2n\to\infty} N^{\lambda} L_{\lambda,\infty,1}(N, A_N) \leqslant \max(|C_1(\lambda) - A|, |A|). \tag{5.2}$$

Further (5.1) implies that

$$\begin{aligned} &\liminf_{N=2n\to\infty} N^{\lambda} L_{\lambda,\infty,1}(N, A_N) \\ &\quad \geqslant \liminf_{N=2n\to\infty} N^{\lambda}|\,|t_0|^{\lambda} - S_{N,1}(t_0) - (-1)^n N^{-\lambda} A_N T_{N,1}(t_0)| \\ &\quad = \max(|C_1(\lambda) - A|, |A|), \end{aligned} \tag{5.3}$$

where $t_0 = 0$ if $|A| > |C_1(\lambda) - A|$, and $t_0 = 1$ otherwise. Then by Lemma 5(a),

$$||f_{\lambda} - g_{\lambda,A,1}||_{L_\infty(\mathbf{R})} = \sup_{t\in\mathbf{R}} |\cos t(F_{\lambda,1}(t) - A)| = \max(|C_1(\lambda) - A|, |A|). \tag{5.4}$$

Thus (5.2)–(5.4) yield (2.1).

(b) Asymptotic (5.1) shows that to prove (2.2), it suffices to find an increasing subsequence $\{2n_j(t)\}_{j=1}^{\infty}$ of indices such that $\lim_{j\to\infty} |T_{2n_j,1}(t)| = 1$, where $t \in [-1, 1]$. If $\alpha := (\arccos t)/\pi$ is a rational number m/k, then it is clear that $n_j = kj,\ j \in \mathbf{N}$. If α is irrational, then the existence of such a sequence follows from the well-known fact that the sequence $\{n\alpha \,(\mathrm{mod}\, 1)\}_{n=0}^{\infty}$ is dense in $[0, 1]$.

(c) By Lemma 3(b), $\cos t\Phi_{\lambda,1}(t) \in L_p(\mathbf{R})$, if $p \in (\frac{1}{2}, \infty)$. Then using asymptotic (2.13) and Lemma 3(c), we obtain

$$\begin{aligned} &\lim_{N=2n\to\infty} N^{\lambda+1/p} L_{\lambda,p,1}(N, A_{N,1}) \\ &\quad = \lim_{N=2n\to\infty} N^{1/p}(1 + O(n^{-1/3}))||T_{N,1}\Phi_{\lambda,1}(N\cdot)||_{L_p[-1,1]} \\ &\quad = ||\cos(\cdot)\Phi_{\lambda,1}(\cdot)||_{L_p(\mathbf{R})} < \infty. \end{aligned} \tag{5.5}$$

Thus (4.14) and (5.5) yield (2.3).

(d) Let $\lim_{n\to\infty} A_{2n} = A$, and let $B_N := (-1)^n(A_N - A_{N,1})$, where $N = 2n,\ n \in \mathbf{N}$. Suppose first $A \neq C_1(\lambda)$ and $p \in (0, \infty)$. Then by Lemma 1(b), we have $\lim_{n\to\infty} B_{4n} \neq 0$. Next, we derive from (2.13) that

$$\lim_{N=2n\to\infty} N^{\lambda+1/p} L_{\lambda,p,1}(N, A_N)$$
$$\geqslant \liminf_{N=4n\to\infty} \left(\int_{-N}^{N} |T_N(t/N)(\Phi_{\lambda,1}(t)(1+\alpha_{N,1,2}(t/N)) - B_N)|^p \, dt \right)^{1/p}. \quad (5.6)$$

Thus (5.6) and Lemma 3(d) yield (2.4). If $A = C_1(\lambda)$ and $p \in (0, \frac{1}{2}]$, then by Lemma 1(b), $\lim_{N=2n\to\infty} B_N = 0$. Now (2.4) follows from Lemma 3(e). The proof of Theorem 1 is completed. ∎

Proof of Theorem 2. The proofs of statements (a) and (b) for $p \neq \infty$ are similar to those of Theorem 1(c) and (d). It remains to prove (2.5) for $p = \infty$.

Using (2.13) and (4.3), we have

$$\limsup_{N=2n\to\infty} N^{\lambda} L_{\lambda,\infty,2}(N, A_{N,2}) \leqslant |\Phi_{\lambda,2}(0)|. \quad (5.7)$$

Next taking account of the relation $\lim_{N=2n\to\infty} T_{N,2}(0) = 1$, we obtain from (1.9) and (1.10)

$$\liminf_{N=2n\to\infty} N^{\lambda} L_{\lambda,\infty,2}(N, A_{N,2})$$
$$\geqslant \liminf_{N=2n\to\infty} N^{\lambda} |f_{\lambda}(0) - S_{N,2}(0) - (-1)^n N^{-\lambda} A_{N,2} T_{N,2}(0)| = |\Phi_{\lambda,2}(0)|. \quad (5.8)$$

Since $|\Phi_{\lambda,2}(0)| = ||\varphi_2 \Phi_{\lambda,2}||_{L_\infty(\mathbf{R})}$, inequalities (5.7) and (5.8) yield (2.5) for $p = \infty$. This completes the proof of Theorem 2. ∎

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